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A Characterization of Finite Auslander–Reiten Quivers

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INTRODUCTION

Let Λ be an indecomposable artin algebra of finite representation type, R the center of Λ, $k = R/\operatorname{rad} R$. Let Γ_Λ be the associated Auslander–Reiten quiver. Then Γ_Λ is finite k-modulated translation quiver and it is unique up to isomorphism of modulated translation quivers (Section 2). The main results of this paper are:

(a) We give necessary and sufficient conditions for a finite k-modulated translation quiver to be an Auslander–Reiten quiver in terms of certain homology groups associated to the quiver (Section 3).

(b) We show that whether a finite k-modulated translation quiver is an Auslander–Reiten quiver depends only on the underlying valued translation quiver and the characteristic of k and give necessary and sufficient conditions only in terms of the valued quiver and char k. We also show that any valued quiver with valuations $(1, n)$ or $(n, 1)$, $n \leqslant 3$, admits a modulation over any prime field (Sections 5, 6, 7).

(c) To a valued translation quiver Γ we associate an unvalued translation quiver $\bar{\Gamma}$ (i.e., all valuations are 1) and show that a finite k-modulated translation quiver is an Auslander–Reiten quiver if and only if the associated unvalued translation quiver with the trivial K-modulation ($K =$ prime field of k) is an Auslander–Reiten quiver (Section 8).

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As a consequence of the last theorem we show that whether a finite k-modulated translation quiver is an Auslander–Reiten quiver depends only on the characteristic of k and furthermore if it is an Auslander–Reiten quiver over some field then it is an Auslander–Reiten quiver for all fields of characteristic 0 and all but a finite number of prime characteristics (Section 9). (R. Bautista proved that it does not depend on the characteristic either [1].)

In Sections 10 and 11 we give the proofs for some results about coverings in general, which are widely accepted as being easily obtainable (proofs for trivially valued quivers are given in [7, 4]).

In Section 12 we give some applications. We show that every finite modulated Auslander–Reiten quiver admits a finite covering which has no short chains. We also show that an automorphism of an artin algebra of finite representation type which fixes an indecomposable module also fixes an indecomposable projective.

In Section 13 we give a short proof for the well-known fact that finite Auslander–Reiten quivers are tree finite. This proof is independent of the rest of the paper.

The writing of this paper was largely inspired by [4]. We use their notation and some of their definitions, however our proofs mostly depend on our results from [5].

1. Modulated Translation Quivers

In this section we develop the basic properties of modulated translation quivers including the concept of homology in such a quiver. (The homology of an ordinary translation quiver has a simple interpretation given in the second and third paragraphs of Section 9.) We start with the definitions.

A *quiver* Q is a directed graph. Thus Q consists of a set of *vertices* Q_0 and a set of *arrows* Q_1 so that each arrow $\alpha \in Q_1$ goes from one element x of Q_0 to one element y of Q_0. We write α: $x \to y$ and we call x a *predecessor* of y and y is called a *successor* of x. We write x^+ for the set of all successors of x and x^- for the set of all predecessors of x.

Definition 1.1. A *translation quiver* Γ is a quiver together with a partially defined function τ: $\Gamma_0 \to \Gamma_0$ so that the following conditions are satisfied:

(1) For any $x, y \in \Gamma_0$ there is at most one arrow $x \to y$ in Γ_1.

(2) For every $x \in \Gamma_0$ the sets x^+ and x^- are finite.

(3) If $\tau x = \tau y$ then $x = y$.

(4) For every vertex x in the domain of τ we have $x^- = (\tau x)^+$.

We will say that a vertex of Γ is *projective* if it is not in the domain of τ and we will call it *injective* if it is not in the range of τ. If $\alpha: x \to y$ is an arrow of Γ where y is not projective then $x \in y^- = (\tau y)^+$, so there is an arrow $\tau y \to x$ which we will call $\sigma\alpha$.

DEFINITION 1.2. Let Γ be a translation quiver and let k be a field. Then a *k-modulation* on Γ consists of the following:

(a) A finite-dimensional division algebra Fx over k for every $x \in \Gamma_0$.

(b) A finite-dimensional Fy–Fx-bimodule ${}_yM_x$ for every arrow $x \to y$. (If there is no arrow $x \to y$ then we set ${}_yM_x = 0$.)

(c) A k-algebra isomorphism $\tau_*: Fx \xrightarrow{\cong} F\tau x$ for each nonprojective vertex x.

(d) A nonsingular bilinear pairing $\sigma_*: {}_yM_{\tau x} \otimes_{Fy} {}_xM_y \to Fy$ for every nonprojective $x \in \Gamma_0$ and $y \in x^-$.

If Γ is a translation quiver and k is a field then we can give Γ the *trivial k-modulation* which is given by: (a) $Fx = k$ for all $x \in \Gamma_0$, (b) ${}_yM_x = k$ for all $x \to y \in \Gamma_1$, (c) τ_* is the identity map, (d) $\sigma_*(a \otimes b) = ab$.

DEFINITION 1.3. Let Γ be a k-modulated translation quiver. Then the *path category* P of Γ is the category whose objects are the vertices of Γ and whose morphism sets $P(x, y)$ are defined as follows.

(1) $P(x, y) = \coprod_{i \geqslant 0} P^i(x, y)$.

(2) $P^0(x, y) = \begin{cases} Fx & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$

(3) $P^i(x, y) = \coprod_{z \in y^-} {}_yM_z \otimes_{Fz} P^{i-1}(x, z)$ if $i \geqslant 1$.

(4) $P^i(x, y) = 0$ if $i < 0$.

Composition of morphisms is given in the obvious way: $f \cdot g = f \otimes g$.

Let $x, y \in \Gamma_0$ where y is not projective. Let $u_1, \ldots, u_d$ be a right Fx-basis for ${}_yM_x$. Since σ_* induces an isomorphism ${}_xM_{\tau y} \cong \operatorname{Hom}_{Fx}({}_yM_x, Fx)$ we get a left dual basis $u_1^*, \ldots, u_d^*$ for ${}_xM_{\tau y}$ so that $\sigma_*(u_i^* \otimes u_j) = \delta_{ij}$.

DEFINITION 1.4. We define the *mesh relation* $\gamma y \in P^2(\tau y, y)$ as follows:

(1) $\gamma y = \sum_{x \in y^-} \gamma y(x)$.

(2) $\gamma y(x) = \sum_{i=1}^d u_i \otimes u_i^* \in {}_yM_x \otimes_{Fx} {}_xM_{\tau y}$.

To show that $\gamma y(x)$ is well defined suppose that $v_1, \ldots, v_d$ is another right Fx-basis for ${}_yM_x$. Then $v_i = \sum_{j=1}^d u_j t_{ji}$, where $(t_{ji}) \in Gl_n(Fx)$. Let

$(s_{ij})=(t_{ji})^{-1}$. Then the dual basis for $\{v_i\}$ is easily seen to be $\{v_i^*\}$, where $v_i^*=\sum_{j=1}^{d} s_{ij}u_j^*$ and we get:

$$\sum v_i\otimes v_i^*=\sum u_j t_{ji}\otimes s_{ik}u_k^*=\sum u_j\otimes t_{ji}s_{ik}u_k^*=\sum u_j\otimes u_j^*.$$

We shall now construct a chain complex $C_*(x,y)$ and consider the homology $H_*(x,y)$ that it defines.

DEFINITION 1.5. For $x,y\in\Gamma_0$ and $m\in\mathbb{Z}$ let $C_m(x,y)$ be the Fy–Fx-bimodule defined as follows.

(1) $C_m(x,y)=\sum_{i\geqslant 2m} C_m^i(x,y)$.

(2) $C_m^i(x,y)=0$ if $m<0$ or $i<2m$.

(3) $C_0^i(x,y)=P^i(x,y)$.

(4) If $m>0$ then $C_m^i(x,y)=\coprod_{z\in y^-} {}_yM_z\otimes_{Fz}C_m^{i-1}(x,z)\amalg Fy\otimes_{Fy} C_{m-1}^{i-2}(x,\tau y)$.

This definition can be reworded as follows:

$$C_m(x,y)=\coprod P(x_m,y)\otimes_{Fx_m}P(x_{m-1},\tau x_m)\otimes_{Fx_{m-1}}\cdots\otimes_{Fx_1}P(x,\tau x_1)\qquad (*)$$

where the direct sum is taken over all m-tuples of nonprojective vertices $x_1,\dots,x_m$.

For $j=1,\dots,m$ let $\partial_m^j\colon C_m(x,y)\to C_{m-1}(x,y)$ be the degree 0 graded map given by $\partial_m^j(f_m\otimes\cdots\otimes f_0)=f_m\otimes\cdots\otimes f_j\cdot\gamma x_j\cdot f_{j-1}\otimes\cdots\otimes f_0$, where we write generators of $C_m(x,y)$ in the form $(*)$. Let $\partial_m=\sum_{j=1}^{m}(-1)^j\partial_m^j$. Then it is clear that $\partial_{m-1}\partial_m=0$ so $(C_*(x,y),\partial_*)$ forms a chain complex. If x,y,z are vertices then we have a map

$$\mu\colon C_*(y,z)\otimes C_*(x,y)\to C_*(x,z)$$

given by sending $(f_m\otimes\cdots\otimes f_0)\otimes(g_n\otimes\cdots\otimes g_0)$ to $f_m\otimes\cdots\otimes f_0\cdot g_n\otimes\cdots\otimes g_0$. Since $\partial\mu(a\otimes b)=\mu(a\otimes b)+(-1)^{\deg b}(\partial a\otimes b)$ we know that μ induces a map on the homology groups.

DEFINITION 1.6. Let $H_*(x,y)$ be the homology of $C_*(x,y)$ and let

$$\mu_*\colon H_*(y,z)\otimes H_*(x,y)\to H_*(x,z)$$

be the map induced by μ.

Since $\mu(\mu\otimes 1)=\mu(1\otimes\mu)$ we get that $\mu_*(\mu_*\otimes 1)=\mu_*(1\otimes\mu_*)$. This means that C_* and H_* are k-categories with Γ_0 being the set of objects in each case.

DEFINITION 1.7. The *mesh category* of Γ is defined to be the category $H_0(\Gamma)$ whose object set is Γ_0 and whose morphism sets are $H_0(x, y)$. Let add $H_0(\Gamma)$ be the additive category generated by $H_0(\Gamma)$. Similarly let $H_*(\Gamma)$ and $C_*(\Gamma)$ be the categories with object sets Γ_0 and morphism sets $H_*(x, y)$ and $C_*(x, y)$.

DEFINITION 1.8. If $x \in \Gamma_0$ let ex^+, ex^-, τx, $\tau^{-1}x$ denote the following objects of add $H_0(\Gamma)$:

(1) $ex^+ = \coprod_{y \in x^+} d_{xy} y$, where $d_{xy} = \dim_{Fy}({}_yM_x)$.

(2) $ex^- = \coprod_{z \in x^-} d'_{zx} z$, where $d'_{zx} = \dim_{Fz}({}_xM_z)$.

(3) $\tau x = 0$ if x is projective.

(4) $\tau^{-1}x = 0$ if x is injective.

If $x \in \Gamma_0$ then let $S_x \colon H_0(\Gamma) \to Ab$ be the contravariant simple functor $S_x = H_0^0(\ ,x) = P^0(\ ,x)$. Let $\bar{C}_*(\ ,\tau x)$ be the chain complex of contravariant functors on $H_0(\Gamma)$ given by $\bar{C}_m(\ ,\tau x) = C_{m-1}(\ ,\tau x)$ if $m \geqslant 1$ and $\bar{C}_0(\ ,\tau x) = S_x$. Let $\bar{\partial}_m \colon \bar{C}_m(\ ,\tau x) \to \bar{C}_{m-1}(\ ,\tau x)$ be the boundary ∂_{m-1} of $C_*(\ ,\tau x)$ if $m > 1$ and let $\bar{\partial}_1 = 0$.

LEMMA 1.9. *There is a short exact sequence of chain complexes of graded functors as follows*:

$$0 \longrightarrow C_*(\ ,ex^-) \xrightarrow{\alpha} C_*(\ ,x) \xrightarrow{\delta} \bar{C}_*(\ ,\tau x) \longrightarrow 0.$$

Furthermore α, δ *are graded maps of degree* 1, -2, *respectively, and* α *is given by composition with some element* α_x *of* $C_0^1(ex^-, x)$.

Proof. We first define α. For each $y_i \in x^-$ let $\{u_{ij}\}$ be a right Fy_i-basis for ${}_xM_{y_i}$. Let $\alpha \colon C_m(\ ,ex^-) \to C_m(\ ,x)$ be given by $\alpha(c_{ij}) = \sum u_{ij} \otimes c_{ij} \in C_m(z, x)$ if $c_{ij} \in C_m(z, y_i)$. Then α is a degree 1 graded chain map given by composition with $\alpha_x = (u_{ij}) \in C_0^1(ex^-, x)$. Note that α_x depends on the choice of basis $\{u_{ij}\}$.

We now define the chain map δ. Let $c \in C_m^t(z, x)$, where $m \geqslant 1$. By 1.5(4) we can express c uniquely in the form

$$c = 1 \otimes e + \sum u_{ij} \otimes c_{ij} \tag{$*$}$$

where $e \in C_{m-1}^{t-2}(z, \tau x)$ and $c_{ij} \in C_m^{t-1}(z, y_i)$. Let $\delta(c) = e$. When $m = 0$ let $\delta \colon C_0(z, x) = P(z, x) \to S_x(z) = P^0(z, x)$ be the projection map. To verify that δ is a chain map we take the boundary of both sides of $(*)$ and get:

$$\partial_m c = \partial_m(1 \otimes e) + \partial_m \left(\sum u_{ij} \otimes c_{ij}\right)$$

$$= 1 \otimes \partial_{m-1} e + (-1)^m \gamma x \otimes e + \sum u_{ij} \otimes \partial_m c_{ij}$$

$$= 1 + \partial_{m-1} e + \sum u_{ij} \otimes [(-1)^m u_{ij}^* \otimes e + \partial_m c_{ij}]$$

where γx is the mesh relation of x and $\{u_{ij}^*\}$ is the dual basis of $\{u_{ij}\}$. Thus $\delta\partial_m c = \partial_{m-1}\delta c = \bar{\partial}_m \delta c$ if $m > 1$ and $\delta\partial_1 c = 0 = \bar{\partial}_1 \delta c$.

The uniqueness of the decomposition $(*)$ means that the following sequence is exact for $m \geqslant 1$:

$$0 \longrightarrow C_m(\ , ex) \xrightarrow{\alpha} C_m(\ , x) \xrightarrow{\delta} \bar{C}_m(\ , \tau x) \longrightarrow 0.$$

The exactness of this sequence for $m = 0$ follows from 1.3. ∎

Since the mth homology of $\bar{C}_*(\ , \tau x)$ is $H_{m-1}(\ , \tau x)$ except when $m = 0$ where it is S_x we get:

PROPOSITION 1.10. *For every $x \in \Gamma_0$ there is a long exact sequence of functors*:

$$\longrightarrow H_{m+1}(\ , x) \xrightarrow{\delta_*} H_m(\ , \tau x) \xrightarrow{\beta_*} H_m(\ , ex) \xrightarrow{\alpha_*} H_m(\ , x) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_0(\ , x) \longrightarrow S_x \longrightarrow 0.$$

Furthermore α_*, β_*, δ_* are graded maps of degree 1, 1, -2, *respectively, and α_*, β_* are given by composition with elements α_x, β_x of $H_0^1(ex^-, x)$, $H_0^1(\tau x, ex^-)$, respectively.*

Proof. Everything follows from 1.9 except for the statement that β_* is given by composition by an element of $H_0^1(\tau x, ex^-)$. Thus we need to identify the connecting homomorphism β_*: $H_m(\ , \tau x) \to H_m(\ , ex^-)$ in 1.10 at the chain level and show that it is given by composition with an element of $C_0^1(\tau x, ex^-)$. Let $c \in C_m(z, \tau x)$ so that $\partial_m c = 0$. If we consider c as an element of $C_{m+1}(z, x)$ we have $\partial_{m+1} c = (-1)^{m+1} \gamma x \otimes c = \sum u_{ij} \otimes u_{ij}^* \otimes (-1)^{m+1} c = \alpha(u_{ij}^* \otimes (-1)^{m+1} c)$. Thus β_* is induced by the chain map β: $C_m(z, \tau x) \to C_m(z, ex^-)$ given by composition with $\beta_x = ((-1)^{m+1} u_{ij}^*) \in C_0^1(\tau x, ex^-)$. (If x is projective then $\beta_x = 0$.) ∎

Proposition 1.10 has a dual statement for covariant functors:

PROPOSITION 1.11. *For every vertex x of Γ we have an exact sequence of covariant functors on $H_0(\Gamma)$:*

$$\cdots \to H_{m+1}(x,\ ) \to H_m(\tau^{-1}x,\ ) \to H_m(ex^+,\ ) \to H_m(x,\ ) \to \cdots$$
$$\cdots \to H_0(x,\ ) \to DS_x \to 0$$

where $DS_x = H_0^0(x,\ )$.

2. MORPHISMS OF MODULATED TRANSLATION QUIVERS

The purpose of this section is to show that the Auslander–Reiten quiver of an artin algebra of finite representation type over an indecomposable artin ring is a modulated translation quiver where the modulation is determined up to isomorphism.

DEFINITION 2.1. Let Γ, Γ' be translation quivers. Then a *translation quiver morphism* $f\colon \Gamma \to \Gamma'$ is a function $f\colon \Gamma_0 \to \Gamma_0'$ satisfying the following:

(i) If x is nonprojective then fx is nonprojective.

(ii) $f\tau x = \tau f x$.

(iii) $f(x^-) \subset (fx)^-$.

DEFINITION 2.2. A *morphism of k-modulated translation quivers* $\Gamma \to \Gamma'$ consists of (a), (b), (c), (d) satisfying conditions (1), (2) below:

(a) $f\colon \Gamma \to \Gamma'$ a translation quiver morphism.

(b) $f_*\colon Fx \to Ffx$ a k-algebra homomorphism for every $x \in \Gamma_0$.

(c) $f_*\colon {}_yM_x \to {}_{fy}M_{fx}$ an Fy–Fx-bimodule morphism for every arrow $x \to y$ in Γ_1.

(d) A nonzero element ϕx of $F\tau f x$ for every nonprojective x in Γ.

Conditions:

(1) If x is a nonprojective vertex of Γ and $g \in Fx$ then $\tau'_* f_*(g) = (\phi x)^{-1} f_* \tau_*(g)(\phi x)$. This means the following diagram commutes:

$$\begin{tikzcd} Fx \arrow[r, "\tau_*"] \arrow[dd, "f_*"'] & F\tau x \arrow[d, "f_*"] \\ & F\tau f x \arrow[d, "\text{conjugation by } \phi x"] \\ Ffx \arrow[r, "\tau'_*"] & F\tau f x \end{tikzcd}$$

(2) If x is a nonprojective vertex of Γ, $y \in x^-$, $u \in {}_yM_{\tau x}$ and $v \in {}_xM_y$, then $f_*\sigma_*(u \otimes v) = \sigma'_*(f_* u(\phi x) \otimes f_* v)$. This means the following diagram commutes:

$$\begin{tikzcd} {}_yM_{\tau x} \otimes_{F x} {}_xM_y \arrow[r, "\sigma_*"] \arrow[d, "f_*(\ )(\phi x) \otimes f_*"'] & Fy \arrow[d, "f_*"] \\ {}_{fy}M_{\tau f x} \otimes_{F f x} {}_{fx}M_{fy} \arrow[r, "\sigma_*"] & Ffy \end{tikzcd}$$

Note that by (c) and (1) the map $f_*(\ )(\phi x)$: ${}_yM_{\tau x} \to {}_{fy}M_{\tau f x}$ is an Fy–Fx-bimodule morphism.

Definition 2.3. Composition of morphisms $\Gamma \to^{f} \Gamma' \to^{f'} \Gamma''$ is defined as follows: $(f', f'_*, \phi')(f, f_*, \phi) = (f'f, f'_* f_*, \phi'')$, where $\phi'' x = f'_*(\phi x)(\phi' f x)$. The *identity* morphism $\Gamma \to \Gamma$ is $(1, 1, 1)$.

Suppose that (f, f_*, ϕ): $\Gamma \to \Gamma'$ is a morphism of k-modulated translation quivers. Then we get an induced functor $H_*(f)$: $H_*(\Gamma) \to H_*(\Gamma')$ defined at the chain level as follows:

$$C_m(f)\colon C_m(x, y) \to C_m(fx, fy).$$

$C_m(f)(c_m \otimes \cdots \otimes c_0) = c_m \otimes (\phi x_m)^{-1}\ c_{m-1} \otimes \cdots \otimes (\phi x_1)^{-1} c_0$ if $c_m \in P(x_m, y)$, $c_0 \in P(x, \tau x_1)$ and $c_i \in P(x_i, \tau x_{i+1})$ for $0 < i < m$. This is a chain map because of the following formula for the mesh relation of fx: $\gamma fx(fy) = \sum u_i \otimes u_i^*(\phi x)^{-1}$ if $\gamma x(y) = \sum u_i \otimes u_i^*$. Note that the identity morphism $\Gamma \to \Gamma$ induces the identity functor $H_*(\Gamma) \to H_*(\Gamma)$.

Definition 2.4. Let Λ be an artin algebra over an indecomposable artin ring R and let $k = R/rR$ (r = radical). Then we define Γ_Λ to be the Auslander–Reiten translation quiver of Λ with the following k-modulation:

(a) $Fx = \operatorname{End}(x)/r \operatorname{End}(x)$.

(b) ${}_yM_x = r(\ , y)/r^2(\ , y)\ (x)$ = the bimodule of irreducible maps from x to y.

(c) For each nonprojective indecomposable Λ-module x we *choose* an almost split sequence $0 \to \tau x \to^{\beta} ex^- \to^{\alpha} x \to 0$. This induces an isomorphism $\tau_*: Fx \to F\tau x$ as indicated in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tau x \arrow[r] \arrow[d, "\tau_* g"'] & ex^- \arrow[r] \arrow[d] & x \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & \tau x \arrow[r] & ex^- \arrow[r] & x \arrow[r] & 0 \end{tikzcd}$$

(d) If $u \in {}_yM_{\tau x}$ and $v \in {}_xM_y$ let $\sigma_*(u \otimes v) = \tilde{u}\tilde{v}$ (modulo r End(y)) as indicated below:

$$\begin{tikzcd}
 & & & y \arrow[dl, dashed, "\tilde{v}"'] \arrow[d, "v"] & \\
0 \arrow[r] & \tau x \arrow[r] \arrow[d, "u"'] & ex^- \arrow[r] \arrow[dl, dashed, "\tilde{u}"] & x \arrow[r] & 0 \\
 & y & & &
\end{tikzcd}$$

The reader can verify that this is a k-modulation. We shall prove:

Proposition 2.5. *Γ_Λ is well defined up to isomorphism of modulated translation quivers.*

Proof. The only ambiguity in the definition of Γ_Λ is the choice of the almost split sequence for x. Suppose that we use another almost split sequence $0 \to \tau x \to^{\beta'} ex^- \to^{\alpha'} x \to 0$. Then for each nonprojective x we get $\phi x \in F\tau x$ given as follows:

$$\begin{tikzcd}
0 \arrow[r] & \tau x \arrow[r, "\beta'"] \arrow[d, "\phi x"'] & ex^- \arrow[r, "\alpha'"] \arrow[d, "\cong"] & x \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \tau x \arrow[r, "\beta"] & ex^- \arrow[r, "\alpha"] & x \arrow[r] & 0
\end{tikzcd}$$

(Even though $\phi x \in \mathrm{End}(\tau x)$ it is uniquely determined only as an element of $F\tau x$.)

The following diagram shows that $\tau'_* g = (\phi x)^{-1} \tau_* g(\phi x)$, where τ'_*: $Fx \to F\tau x$ is the isomorphism induced by the new almost split sequence:

$$\begin{tikzcd}
\tau x \arrow[rr, "\beta'"] \arrow[dr, "\phi x"'] \arrow[dd, "\tau'_* g"'] & & ex^- \arrow[rr, "\alpha'"] \arrow[dr] \arrow[dd] & & x \arrow[dr] \arrow[dd] & \\
 & \tau x \arrow[rr, crossing over, "\beta"] & & ex^- \arrow[rr, crossing over, "\alpha"] & & x \arrow[dd, "g"] \\
\tau x \arrow[rr, "\beta'"] \arrow[dr, "\phi x"'] & & ex^- \arrow[rr, "\alpha'"] \arrow[dr] & & x \arrow[dr] & \\
 & \tau x \arrow[rr, "\beta"'] \arrow[from=uu, crossing over, "\tau_* g"] & & ex^- \arrow[rr, "\alpha"'] \arrow[from=uu, crossing over] & & x
\end{tikzcd}$$

Let $u \in {}_yM_{\tau x}$ and $v \in {}_xM_y$. Then $\sigma'_*(u \otimes v) = \sigma_*(u(\phi x)^{-1} \otimes v)$ as indicated in the following diagram:

$$\begin{tikzcd}
 & & y \arrow[dl, dashed] \arrow[d, "v"] \arrow[dd, bend left=60, "v"] \\
\tau x \arrow[r, "\beta'"] \arrow[d, "\phi x"] \arrow[dd, bend right=60, "u"'] & ex^{-} \arrow[r, "\alpha'"] \arrow[d] & x \arrow[d] \\
\tau x \arrow[r, "\beta"] \arrow[d, "u(\phi x)^{-1}"] & ex^{-} \arrow[r, "\alpha"] \arrow[dl, dashed] & x \\
y & &
\end{tikzcd}$$

These formulas for τ'_* and σ'_* show that $(1, 1, \phi)$ is a morphism $\Gamma_\Lambda \to \Gamma'_\Lambda$. This is an isomorphism with inverse $(1, 1, \phi^*)$, where $\phi^* x = (\phi x)^{-1}$. ∎

In general we have the following:

LEMMA 2.6. *Let (f, f_*, ϕ): $\Gamma \to \Gamma'$ be a morphism of k-modulated translation quivers. Then this is an isomorphism if and only if f, f_* are isomorphisms.*

Proof. $(f, f_*, \phi)^{-1} = (f^{-1}, f_*^{-1}, \phi^*)$, where $\phi^* x = f_*^{-1}(\phi f^{-1} x)^{-1}$. ∎

PROPOSITION 2.7. *Let Γ be a tree finite translation quiver and let k be a field. Then any k-modulation on Γ which makes all division algebras and bimodules one dimensional over k is isomorphic to the trivial k-modulation on Γ.*

Proof. This is a rewording of [4, Lemma 5.1]. If we choose isomorphisms $Fx \cong k$ and ${}_yM_x \cong k$ then the maps τ_*, σ_* are given by $\tau_*(a) = a$ and $\sigma_*(a \otimes b) = z_\beta ab$ if $a \in {}_yM_{\tau x}$, $b \in {}_xM_y$, β: $y \to x \in \Gamma_1$ and $z_\beta \in k^* = k - \{0\}$.

Let b_γ be the nonzero element of k given by [4, Lemma 5.1]. If Γ' is Γ with the trivial k-modulation we get an isomorphism $(1, f_*, \phi)$: $\Gamma \to \Gamma'$, where f_* and ϕ are given as follows:

(1) $f_*: Fx \to Fx$ is the identity map for every $x \in \Gamma_0$.

(2) $f_*: {}_yM_x \to {}_yM_x$ is given by $f_*(a) = b_\beta a$ for every β: $x \to y \in \Gamma_1$.

(3) $\phi x = b_\gamma x$ for each nonprojective $x \in \Gamma_0$. ∎

3. THE CONDITIONS

The purpose of this section is to characterize finite modulated translation quivers which are Auslander–Reiten quivers. This characterization will be used later for deriving the characterization of Auslander–Reiten quivers considered only as valued quivers. Results about coverings will also follow easily from this characterization.

LEMMA 3.1. *Let Γ_Λ be an Auslander–Reiten quiver of an artin algebra of finite representation type. Let $x \in \Gamma_0$. Then $H_0(\ ,x) \cong G(\ ,x)$ as graded functors, where $H_0(\ ,x) = \coprod_{j \geqslant 0} H_0^j(\ ,x)$ and $G(\ ,x) = \coprod_{j \geqslant 0} r^j(\ ,x)/r^{j+1}(\ ,x)$.*

Proof. Let $h: H_0(\ ,x) \to G(\ ,x)$ be the degree 0 map given by composition of irreducible maps. Then $h_0: H_0^0(\ ,x) \to G_0(\ ,x)$ and $h_1: H_0(\ ,x) \to G(\ ,x)$ are isomorphisms by definition. Assume by induction that $j \geqslant 2$ and h_{j-1}, h_{j-2} are isomorphisms. Then the five lemma applied to the following diagram shows that h_j is an isomorphism.

$$\begin{tikzcd}
H_1^j(\ ,x) \arrow[r, "\delta_*"] \arrow[d] & H_0^{j-2}(\ ,\tau x) \arrow[r] \arrow[d, "\cong"', "h_{j-2}"] & H_0^{j-1}(\ ,ex^-) \arrow[r] \arrow[d, "\cong"', "h_{j-1}"] & H_0^j(\ ,x) \arrow[r] \arrow[d, "h_j"] & 0 \\
0 \arrow[r] & G_{j-2}(\ ,\tau x) \arrow[r] & G_{j-1}(\ ,ex^-) \arrow[r] & G_j(\ ,x) \arrow[r] & 0
\end{tikzcd}$$

The exactness of the top row is given in 1.10. The exactness of the bottom row is proved in [5]. ∎

COROLLARY 3.2. *The map $\delta_*: H_1^j(\ ,x) \to H_0^{j-2}(\ ,\tau x)$ is zero.*

LEMMA 3.3. *Let A be an artin algebra with* $\operatorname{gl\,dim} A \leqslant 2$. *Then A is an Auslander algebra if and only if every simple of projective dimension $\leqslant 1$ is a submodule of an injective projective.*

Proof. ($\Rightarrow$) Let A be the Auslander algebra of Λ. Then every simple A-module of projective dimension $\leqslant 1$ has the form $S_p = (\ ,p)/r(\ ,p)$ for some projective Λ-module p. But $S_p \hookrightarrow (\ ,i(p/rp))$, where $i(p/rp)$ is the injective envelope of p/rp and $(\ ,i)$ is an injective A-module for every injective Λ-module i.

($\Leftarrow$) Since A is projective $\operatorname{pr\,dim} \operatorname{soc} A \leqslant 1$. Thus by assumption the injective envelope I_0 of $\operatorname{soc} A$ is projective. Let $L = \operatorname{Coker}(A \to I_0)$. Then $\operatorname{pr\,dim} L \leqslant 1$ so $\operatorname{pr\,dim} \operatorname{soc} L \leqslant 1$. Thus the injective envelope of L is projective. This proves that $\operatorname{dom\,dim} A \geqslant 2$. So A is an Auslander algebra. ∎

THEOREM 3.4. *A finite nonempty k-modulated translation quiver Γ is isomorphic to the Auslander–Reiten quiver of an artin algebra of finite representation type if and only if it satisfies the following conditions:*

(a) *$H_0(x, y)$ is finite dimensional over k for all $x, y \in \Gamma_0$.*

(b) *$H_1(x, y) = 0$ for all $x, y \in \Gamma_0$.*

(c) *If p is a projective vertex and i is an injective vertex then the map $\alpha^*\colon H_0(p, i) \to H_0(ep^-, i)$ is onto.*

Proof. ($\Rightarrow$)(a) From Lemma 3.1 we have $H_0(\ ,x) \cong G(\ ,x)$ and since $G(\ ,x)$ has the same length as $(\ ,x)$ it follows that $H_0(\ ,x)$ has finite length.

(b) Proof is by induction. We know that $H_1^j(\ ,y) = 0$ for $j \leqslant 1$. So suppose that $j \geqslant 2$ and $H_1^{j-1}(\ ,y) = 0$ for all $y \in \Gamma_0$. Then the exactness of the following sequence and the fact that $\delta_* = 0$ (3.2) imply that $H_1^j(\ ,x) = 0$.

$$H_1^{j-1}(\ ,ex^-) \longrightarrow H_1^j(\ ,x) \xrightarrow{\delta_*} H_0^{j-2}(\ ,x).$$

(c) It follows from [5] that for every projective p the following sequence is exact:

$$0 \to G(p/rp,\ ) \to G(p,\ ) \to G(rp,\ ).$$

When we evaluate this sequence on an injective i and compute lengths we see that $lG(p,i) = l(p,i) = l(rp,i) + l(p/rp,i) = lG(rp,i) + lG(p/rp,i)$. Thus $G(p,i)$ maps onto $G(rp,i)$. This implies that $\alpha^*\colon H_0(p,i) \to H_0(ep^-,i)$ is onto.

($\Leftarrow$) Suppose Γ is a finite nonempty k-modulated translation quiver satisfying conditions (a), (b), (c). Let w be the object of add $H_0(\Gamma)$ given by $w = \coprod_{x \in \Gamma_0} x$. Let $A = \operatorname{End}(w) = H_0(w,w) = \coprod H_0(x,y)$. Then we shall show that A is the Auslander algebra of an artin algebra Λ and that $\Gamma \cong \Gamma_\Lambda$.

Condition (a) implies that A is a finite-dimensional algebra over k. Also $H_0(x,x)$ is a local ring for every $x \in \Gamma_0$. Thus $P_x = H_0(w,x)$ are the indecomposable projective A-modules and $P_x/rP_x = S_x$ are simple A-modules.

Condition (b) implies that the following sequence is exact for every $x \in \Gamma_0$:

$$0 \longrightarrow H_0(w, \tau x) \xrightarrow{\beta^*} H_0(w, ex^-) \xrightarrow{\alpha^*} P_x \longrightarrow S_x \longrightarrow 0. \qquad (*)$$

Thus A has global dimension $\leqslant 2$. Note that $(*)$ gives the minimal projective resolution of S_x since $\operatorname{Im} \alpha_* = rP_x$ and $\operatorname{Im} \beta_* \subset rH_0(w, ex^-) = \coprod_{j \geqslant 1} H_0^j(w, ex^-)$. Thus pr dim $S_x = 2$ if and only if x is not projective.

We shall now show that, given (c), each projective vertex p has associated with it an injective vertex i so that P_i is an injective projective with $\operatorname{soc} P_i \cong S_p$. To show that P_i is injective we show that $\operatorname{Ext}_A^1(S_x, P_i) = 0$ for all simple S_x.

If x is not projective we have an exact sequence:

$$0 \longrightarrow H_0(x, y) \xrightarrow{\alpha_*} H_0(ex^-, y) \xrightarrow{\beta_*} H_0(\tau x, y) \longrightarrow S_{\tau x}(y) \longrightarrow 0.$$

Exactness of this sequence at $H_0(ex^-, y)$ implies that $\operatorname{Ext}^1_A(S_x, P_y) = 0$. (Map the sequence $(*)$ above into P_y.)

If x is projective we map the sequence $(*)$ into P_i and get

$$H_0(x, i) \xrightarrow{\alpha^*} H_0(ex^-, i) \longrightarrow \operatorname{Ext}^1_A(S_x, P_i) \longrightarrow 0.$$

By condition (c) we get $\operatorname{Ext}^1_A(S_x, P_i) = 0$. Thus P_i is injective for every injective vertex i.

Since P_i is an indecomposable injective projective its socle is simple with projective dimension $\leqslant 1$. Also distinct i give distinct $\operatorname{soc} P_i$. So $S_p = \operatorname{soc} P_i$ gives a 1–1 correspondence between the projective vertices p and the injective vertices i. By Lemma 3.3 this implies that A is an Auslander algebra.

If Λ is the artin algebra of which A is the Auslander algebra then Γ_Λ is given by $F'x$, ${}_yM'_x$, τ'_*, σ'_*, where $F'x = \operatorname{End}(P_x)/r\operatorname{End}(P_x) \cong Fx$, ${}_yM'_x = \operatorname{Hom}_A(P_x, rP_y/r^2P_y) \cong {}_yM_x$ and τ'_*, σ'_* are given by $(*)$. We get that τ'_*, σ'_* correspond to τ_*, σ_* under the above isomorphism so $\Gamma_\Lambda \cong \Gamma$. ∎

Remarks 3.5. (1) The conditions in Theorem 3.4 are sufficient but not necessary for the mesh category $H_0(\Gamma)$ to be an Auslander category since condition (b) forces x^- to be nonempty for nonprojective x.

(2) Condition 3.4(b) is equivalent to the following:

(b′) $\beta_*\colon H_0(x, \tau y) \to H_0(x, ey^-)$ is a monomorphism for all $x, y \in \Gamma_0$.

(3) Condition 3.4(c) can be replaced by the following:

(c′) $\beta_*\colon H_0(p, i) \to H_0(p, ei^+)$ is onto if p is projective and i is injective.

Proof (2). (b) ⇒ (b′) is clear. (b′) ⇒ (b) follows by induction on i and the following exact sequence:

$$H^i_1(x, ey^-) \longrightarrow H^{i+1}_1(x, y) \longrightarrow H^{i-1}_0(x, \tau y) \xrightarrow{\beta_*} H^i_0(x, ey^-). \quad ∎$$

In order to prove Remark (3) we will develop some basic properties of the opposite quiver of a modulated translation quiver.

DEFINITION 3.6. Let Γ be a translation quiver. Then Γ^{op} = the *opposite* of Γ is defined as follows:

(1) The vertices of Γ^{op} are the vertices of Γ with stars $(*)$. Thus if $x \in \Gamma_0$ then $x^* \in \Gamma_0^{\mathrm{op}}$.

(2) $x^* \to y^* \in \Gamma_1^{\mathrm{op}}$ if and only if $y \to x \in \Gamma_1$.

(3) x^* is projective if and only if x is injective.

(4) $\tau(x^*) = (\tau^{-1}x)^*$ if x^* is not projective.

DEFINITION 3.7. Let Γ be a k-modulated translation quiver. Then the opposite Γ^{op} of Γ is defined to be the opposite of the underlying translation quiver of Γ with the following k-modulation:

(a) $Fx^* = (Fx)^{\mathrm{op}}$ for every $x^* \in \Gamma_0^{\mathrm{op}}$.

(b) ${}_{y^*}M_{x^*} = {}_xM_y$ for every arrow $x^* \to y^* \in \Gamma_1^{\mathrm{op}}$.

(c) $\tau'_*\colon Fx^* \to F\tau x^*$ is given by $\tau'_* = \rho_{Fy}\tau_*^{-1}\rho_{Fx}^{-1}$, where $y = \tau^{-1}x$ and $\rho_{Fx}\colon Fx \to Fx^{\mathrm{op}}$ is the identity anti-isomorphism for every nonprojective $x^* \in \Gamma_0^{\mathrm{op}}$.

(d) $\sigma'_*\colon \ {}_{y^*}M_{\tau x^*} \otimes_{Fx^*} {}_{x^*}M_{y^*} \to Fy^*$ is given by $\sigma'_*(a \otimes b) = \rho_{Fy}\sigma_*(b \otimes a)$ for every $y^* \to x^* \in \Gamma_1^{\mathrm{op}}$, where x^* is not projective.

PROPOSITION 3.8. *Let Λ be an artin algebra of finite representation type over an artin ring R. Then $(\Gamma_\Lambda)^{\mathrm{op}} \cong \Gamma_{\Lambda^{\mathrm{op}}}$.*

Proof. The isomorphism is given by the duality D: $\operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$ given by $D(\ ) = \operatorname{Hom}_R(\ , I)$, where I is the R-injective envelope of R/rR. ∎

Proof of 3.5(3). By 3.8 Γ is an Auslander–Reiten quiver if and only if Γ^{op} is. Since $H_*(x^*, y^*) \cong H_*(y, x)$ we get that Γ^{op} satisfies 3.4(a), (b), (c) if and only if Γ satisfies 3.4(a), (b) and 3.5(c'). ∎

4. FINITENESS OF THE CONDITIONS

In Section 3 we characterized Auslander–Reiten quivers in terms of H_0 and H_1. In this section we shall give the characterization in terms of H_0^j and H_0^j, where $0 \leqslant j \leqslant 4n$, n being the number of vertices in the quiver.

LEMMA 4.1. *Let Γ be a finite k-modulated translation quiver with n vertices where $n \geqslant 1$ and suppose that Γ satisfies* 3.4(b) and (c). *Then the following are equivalent*:

(a) *$H_0(x, y)$ is finite dimensional over k for all $x, y \in \Gamma_0$.*

(a_f) *$H_0^{4n-1}(x, y) = 0$ for all $x, y \in \Gamma_0$.*

Proof. Since $H_0^j(x, ey^-)$ maps onto $H_0^{j+1}(x, y)$ for $j \geqslant 0$ it is clear that (a_f) implies (a). Conversely suppose that Γ satisfies (a). Then Γ is an

Auslander–Reiten quiver and $H_0(x, y) = G(x, y)$. Thus by [5, 7.10] we have (a_f). ▮

THEOREM 4.2. *Let Γ be a finite k-modulated translation quiver with n vertices where $n \geqslant 1$. Then Γ is an Auslander–Reiten quiver if and only if the following conditions are satisfied*:

(a_f) $H_0^{4n-1}(x, y) = 0$ *for all* $x, y \in \Gamma_0$.

(b_f) $H_1^j(x, y) = 0$ *if* $x, y \in \Gamma_0$ *and* $j \leqslant 4n$.

(c_f') $\beta_*: H_0^j(p, i) \to H_0^{j+1}(p, ei^+)$ *is onto for all* $j \leqslant 4n-2$ *if p is projective and i is injective.*

Proof. If Γ is an Auslander–Reiten quiver then by Theorem 3.4, 3.5(3) and 4.1 we know that (a_f), (b_f), (c_f') are satisfied. Conversely suppose that these conditions hold for Γ. Then we shall prove 3.4(a), (b) and 3.5(c′). Condition 3.5(c′) follows immediately from (a_f) and (c_f'). To prove 3.4(b) suppose that $j \geqslant 4n$ and $H_1^j(x, y) = 0$ for all $x, y \in \Gamma_0$. Then (a_f) and the following exact sequence prove that $H_1^{j+1}(x, y) = 0$ for all $x, y \in \Gamma_0$:

$$H_1^j(x, ey^-) \to H_1^{j+1}(x, y) \to H_0^{j-1}(x, \tau y).$$

This proves 3.4(b). Finally, (a_f) obviously implies (a). ▮

5. VALUED TRANSLATION QUIVERS

We consider valued translation quivers where the arrows have valuations $(1, d)$ or $(d, 1)$, where $d = 1$, 2 or 3, since these are the only valuations which can occur in Auslander–Reiten quivers of algebras of finite representation type. We show that any such valued translation quiver admits a modulation over any prime field.

DEFINITION 5.1. Let Γ be a translation quiver. Then a *valuation* on Γ consists of a positive integer m_x associated to each vertex x and a pair of positive integers (d_{xy}, d'_{xy}) associated to each arrow $x \to y$ so that:

(1) $m_{\tau x} = m_x$ for every nonprojective $x \in \Gamma_0$.

(2) $d_{xy} m_y = m_x d'_{xy}$ for every arrow $x \to y$ of Γ.

(3) $d'_{xy} = d_{\tau yx}$ for every arrow $x \to y$ with y nonprojective.

DEFINITION 5.2. If Γ is a k-modulated translation quiver then the *underlying valued translation quiver* of Γ is Γ with the following valuation:

(1) $m_x = \dim_k(F_x)$.

(2) $d_{xy} = \dim_{Fy}({}_yM_x)$.

(3) $d'_{xy} = \dim_{Fx}({}_yM_x)$.

DEFINITION 5.3. Let Γ be a valued translation quiver. We say that Γ satisfies *condition D* if the valuations on the arrows have the form $(1, d)$ or $(d, 1)$, where $d = 1$, 2 or 3.

THEOREM 5.4. *Suppose that Γ is a valued translation quiver satisfying condition D and let K be any prime field ($K = F_p$ or $\mathbb{Q}$). Then Γ admits a K-modulation.*

Proof. For every positive integer m let K_m be the field extension of K of degree m given as follows. If $K = F_p$ then let $K_m = F_{p^m}$. If $K = \mathbb{Q}$ then let $K_m = \mathbb{Q}(2^{1/m})$. Note that if n divides m then K_n is a subfield of K_m of index m/n.

For each vertex x of Γ let $Fx = K_{m_x}$. For each arrow $x \to y$ of Γ let ${}_yM_x = K_m$, where $m = \max(m_x, m_y)$. This is an Fy–Fx-bimodule by multiplication. For each nonprojective vertex x of Γ let $\tau_* : Fx \to F\tau x$ be the identity map. It remains to define σ_*.

Suppose that $x \to y$ is an arrow of Γ where y is not projective. If $d_{xy} = 1$ then ${}_xM_{\tau y} = {}_yM_x = Fx = K_{m_x}$ and $Fy \subset Fx$. Thus we can define σ_*: ${}_xM_{\tau y} \otimes_{Fy} {}_yM_x \to Fx$ by $\sigma_*(a \otimes b) = ab$. If $d_{xy} \neq 1$ then ${}_xM_{\tau y} = {}_yM_x = Fy \supset Fx$ so we can let π: $Fy \to Fx$ be any nonzero Fx-linear map and define $\sigma_*(a \otimes b) = \pi(ab)$. ∎

6. THE DIAGRAM

Let Γ be a valued translation quiver. Then we will associate to each $x \in \Gamma_0$ a simply connected translation quiver Σ_x and a translation quiver morphism f: $\Sigma_x \to \Gamma$ so that $H_0(y, x)$ can be reconstructed from Fy and Σ_x. We will call Σ_x the *diagram* of x. (See [5, Section 1]. There is a change in notation: $\mathscr{V}_i = \mathscr{X}_i$ and $\mathscr{A}_i = \mathscr{F}_i$.)

The set of vertices of Σ_x will be a disjoint union $\mathscr{V} = \coprod_{i \geqslant 0} \mathscr{V}_i$ so that $\tau\mathscr{V}_i \subset \mathscr{V}_{i+2}$. The set of arrows of Σ_x will be a disjoint union $\mathscr{A} = \coprod_{i \geqslant 1} \mathscr{A}_i$ so that the arrows of $\mathscr{A}_i$ go from elements of $\mathscr{V}_i$ to elements of $\mathscr{V}_{i-1}$. As in [5, Section 1], we shall construct the sets $\mathscr{V}_i$ and $\mathscr{A}_i$ together with certain subsets $\mathscr{V}'_i$ and $\mathscr{A}'_i$ and functions $f_i: \mathscr{V}_i \to \Gamma_0$, $f'_i : \mathscr{V}'_i \to \Gamma_0$ by induction on i.

Let $\mathscr{V}_0$ be the set which contains d_{xy} copies of each $y \in x^+$. Let f_0: $\mathscr{V}_0 \to \Gamma_0$ be the map which sends each copy of y to y.

Let $\mathscr{V}'_1 = \{x\}$ and let $f'_i : \mathscr{V}'_1 \to \Gamma_0$ be the inclusion map. Let $\mathscr{A}'_1$ be the set containing one arrow from $x \in \mathscr{V}'_1$ to each element of $\mathscr{V}_0$. Then we have the following for $i = 1$.

(a_i) If $a \to b$ is an arrow in $\mathscr{A}'_i$ then $f_{i-1}(b) \in f'_i(a)^+$.

(b_i) If $a \in \mathscr{T}'_i$ then $a^+ \subset \mathscr{T}_{i-1}$ contains exactly d_{yz} copies of each $z \in y^+$, where $y = f'_i(a)$.

(c_i) For each $b \in \mathscr{T}_{i-1}$ and each $y \in z^-$, where $z = f_{i-1}(b)$, let $h(y, b)$ be the number of arrows $a \to b$ in $\mathscr{A}'_i$ so that $y = f'_i(a)$. Then $h(y, b) \geqslant d'_{yz}$.

Suppose by induction that $\mathscr{T}_{i-1}, f_{i-1}, \mathscr{T}'_i, \mathscr{A}'_i, f'_i$ are defined satisfying (a_i), (b_i) and (c_i). Then we shall construct $\mathscr{T}_i, f_i, \mathscr{T}'_{i+1}, \mathscr{A}'_{i+1}$ and f'_{i+1} so that (a_{i+1}), (b_{i+1}) and (c_{i+1}) are satisfied.

If $b \in \mathscr{T}_{i-1}$ and $f_{i-1}(b) = z$ then let $\mathscr{T}(b)$ be the set containing $d'_{yz} - h(y, b)$ copies of y for each $y \in z^-$. Let $\mathscr{T}_i$ be the disjoint union of $\mathscr{T}'_i$ and the sets $\mathscr{T}(b)$ for all $b \in \mathscr{T}_{i-1}$. Let $f_i: \mathscr{T}_i \to \Gamma_0$ be given by $f_i(a) = f'_i(a)$ if $a \in \mathscr{T}'_i$ and $f_i(a) = y$ if $a \in \mathscr{T}(b)$. For each $b \in \mathscr{T}_{i-1}$ let $\mathscr{A}(b)$ be the set consisting of one arrow from each element of $\mathscr{T}(b)$ to b. Let $\mathscr{A}_i$ be the disjoint union of $\mathscr{A}'_i$ and the sets $\mathscr{A}(b)$ for all $b \in \mathscr{T}_{i-1}$. Let $\mathscr{T}'_{i+1}$ be the set consisting of one copy of τz for each copy of each nonprojective $z \in \Gamma_0$ which occurs in $\mathscr{T}_{i-1}$. Let $f'_{i+1}: \mathscr{T}'_{i+1} \to \Gamma_0$ be the map which sends each copy of τz to τz. Let $\mathscr{A}'_{i+1}$ be the set containing one arrow from each $\tau b \in \mathscr{T}'_{i+1}$ to each element of $b^- \subset \mathscr{T}_i$.

Conditions (a_{i+1}) and (b_{i+1}) are obvious from the construction and the equation $d_{\tau zy} = d'_{yz}$ (7.1(3)). We will verify condition (c_{i+1}). Let $b \in \mathscr{T}_i$, $z = f_i(b)$ and $y \in z^-$. If $b \notin \mathscr{T}'_i$ then $b \in \mathscr{T}(c)$ for some $c \in \mathscr{T}_{i-1}$. This implies that $h(y, b) = 1$ or 0 depending on whether $y = \tau f_{i-1}(c)$ or not. If $b \in \mathscr{T}'_i$ then $h(y, b) = 0$ or d'_{yz} depending on whether y is injective or not. (Use condition (b_i) and the equation $d'_{\tau wz} = d_{zw}$.)

Let Γ be a k-modulated translation quiver which satisfies condition D (i.e., bimodules are one dimensional on one side and at most three dimensional on the other side). Let Σ be the diagram of any vertex x of Γ and let $f: \Sigma_0 = \mathscr{T} \to \Gamma_0$ be the function defined above. Then f is a translation quiver morphism and we have the following lemma.

LEMMA 6.1. *For each arrow α: $a \to b$ in Σ we can choose an $f_*(\alpha) \in {}_{fb}M_{fa}$ so that*

(1) *If $\{a_i\} = b^- \cap f^{-1}(y)$ and α_i: $a_i \to b$ are arrows of Σ then $\{f_*(\alpha_i)\}$ forms a right Fy basis for ${}_{fb}M_y$.*

(2) *$\sigma_*(f_*(\sigma\alpha_i) \otimes f_*(\alpha_j)) = \delta_{ij} \in Fy$ if b is not projective.*

(3) *If $\{b_j\} = a^+ \cap f^{-1}(z)$ and β_j: $a \to b_j$ are arrows of Σ then $\{f_*(\beta_j)\}$ is an Fz-independent set in ${}_zM_{fa}$.*

Proof. This follows from the proof of [5, Theorem 1.15]. We summarize the argument.

For each $y \in x^+$ we choose a "good" left Fy-basis for ${}_yM_x$. (This means a left basis $\{u_1, ..., u_n\}$ for ${}_yM_x$ so that either $n = 1$ or the dual elements

$\{u_1^*, \ldots, u_n^*\}$ form a right Fy-basis for $\mathrm{Hom}_{Fx}({}_yM_x, Fx)$. When $n \neq 1$ we have ${}_yM_x \cong Fx$ by condition D so the *dual elements* can be defined by $u_i^*(u_i) = 1$.) Let f_* map the arrows of $\mathscr{A}_1'$ to these basis elements.

Suppose by induction that f_* is defined on $\mathscr{A}_i', \mathscr{A}_{i-1}', \mathscr{A}_{i-2}', \ldots$. Then by [5, 1.11] we can extend f_* to $\mathscr{A}_i$ so that condition (1) is satisfied in a strong way ($\{f_*(a_i)\}$ will be a good right Fy-basis for ${}_{fb}M_y$). Condition (2) forces our choice of f_* on $\mathscr{A}_{i+1}'$ but we still have good basis by [5, Lemma 1.9], which says that the dual of a good basis is a good basis.

Condition (3) follows from (2) and the choice of f_* on $\mathscr{A}_1'$. If a is not injective in (3) then (3) follows from (2). If a is injective then either $a = x =$ the unique element of $\mathscr{T}_1'$ or a^+ has only one element. ∎

Let Γ be a k-modulated translation quiver satisfying condition D and let K be the prime field of k. Let $x \in \Gamma_0$ and let Σ be the diagram of x. Give Σ the trivial K-modulation and let $f\colon \Sigma_0 \to \Gamma_0$ and f_* be as above.

THEOREM 6.2. (1) *Let $a \in \Sigma_0$ and $y \in \Gamma_0$. Then there is an isomorphism of Fy-vector spaces as follows where $f^{-1}y$ is the direct sum of all the elements of $f^{-1}(y)$:*

$$\psi_i\colon H_0^i(f^{-1}y, a) \otimes_K Fy \to H_0^i(y, fa).$$

(2) *Let $\alpha\colon a \to b$ be an arrow of Σ and let $f_*(\alpha)\colon fa \to fb$ be the corresponding element of ${}_{fb}M_{fa} = H_0^1(fa, fb)$. Then for every $y \in \Gamma_0$ and $i \in \mathbb{Z}$ the following diagram commutes:*

$$\begin{tikzcd} H_0^{i-1}(f^{-1}y, a) \otimes_K Fy \arrow[r, "\psi_{i-1}", "\cong"'] \arrow[d, "\alpha_* \otimes 1"'] & H_0^{i-1}(y, fa) \arrow[d, "f_*(\alpha)_*"] \\ H_0^i(f^{-1}y, b) \otimes_K Fy \arrow[r, "\psi_i", "\cong"'] & H_0^i(y, fb) \end{tikzcd}$$

Proof. By induction on i. It is clear that (1) and (2) are satisfied when $i \leqslant 0$. The map ψ_0 sends $K \otimes_K Fy = Fy$ to Fy by the identity map.

Let $i \geqslant 0$ and suppose that ψ_i is defined satisfying (1) and (2). Then we can define ψ_{i+1} to be the induced map in the following diagram:

$$\begin{tikzcd} H_0^{i-1}(f^{-1}y, \tau a) \otimes Fy \arrow[r, "\beta_*"] \arrow[d, "\psi_{i-1}"] & H_0^i(f^{-1}y, ea^-) \otimes Fy \arrow[r, "\alpha_*"] \arrow[d, "\psi_i"] & H_0^{i+1}(f^{-1}y, a) \otimes Fy \arrow[r] \arrow[d, "\psi_{i+1}"] & 0 \\ H_0^{i-1}(y, \tau fa) \arrow[r, "f_*(\beta)_*"] & H_0^i(y, efa^-) \arrow[r, "f_*(\alpha)_*"] & H_0^{i+1}(y, fa) \arrow[r] & 0 \end{tikzcd}$$

Commutativity of the left-hand square follows from (2). The new function ψ_{i+1} is an isomorphism by the five lemma. To see that ψ_{i+1} satisfies (2) note that $\alpha\colon ea^- \to a$ is the direct sum of all arrows in Σ with target a. ∎

7. Conditions A_p, B_p and C_p

Let Γ be a finite nonempty k-modulated translation quiver and let p be the characteristic of k. Then we shall show that Γ is an Auslander–Reiten quiver if and only if the underlying valued translation quiver of Γ satisfies certain conditions which depend on p.

We now consider Γ to be valued translation quiver and we let K denote the prime field of characteristic p. For each $x \in \Gamma_0$ let Σ_x be the diagram of x. Let $K(\Sigma_x)$ be the mesh category of Σ_x with the trivial K-modulation.

Definition 7.1. We say that Γ satisfies A_p, B_p, C_p, respectively, if the following conditions are satisfied for all $x \in \Gamma_0$:

A_p: $H_0^j(\ ,x) = 0$ as a functor on $K(\Sigma_x)$ for sufficiently large j.

B_p: $\beta_*: H_0^j(\ ,x) \to H_0^{j+1}(\ ,ex^+)$ is a monomorphism of functors on $K(\Sigma_x)$ if x is not injective in Γ.

C_p: $\beta_*: H_0^j(p,i) \to H_0^{j+1}(p,ei^+)$ is an epimorphism in $K(\Sigma_i)$ if i is an injective vertex of Γ and p is a projective vertex of Σ_i.

Theorem 7.2. *Let Γ be a finite nonempty k-modulated translation quiver and let $p =$ char k. Then Γ is the Auslander–Reiten quiver of an artin algebra of finite representation type if and only if the underlying valued quiver of Γ satisfies A_p, B_p, C_p and D.*

Proof. It is well known [2] that every finite Auslander–Reiten quiver satisfies condition D. Given that Γ satisfies D it follows from 6.2(1) that A_p is equivalent to 3.4(a). Also 6.2(2) implies that B_p is equivalent to 3.4(b). Theorem 6.2 also implies that C_p is equivalent to 3.5(c′), which in the presence of 3.4(a) and (b) is equivalent to 3.4(c). ∎

8. The Associated Unvalued Translation Quiver

Suppose that Γ is a finite nonempty valued translation quiver which satisfies condition D (i.e., valuations on arrows are either $(1, d)$ or $(d, 1)$ where $d \leqslant 3$). Then we associate to Γ an unvalued translation quiver $\bar{\Gamma}$ so that Γ is the Auslander–Reiten quiver of an artin algebra of finite representation type if and only if $\bar{\Gamma}$ is a Riedtmann quiver [4]. More precisely we take the *trivial valuation* on $\bar{\Gamma}$ (1 on each vertex and $(1, 1)$ on each arrow) and prove that Γ satisfies A_p, B_p and C_p if and only if $\bar{\Gamma}$ does.

We now give the construction of $\bar{\Gamma}$. Let m be the least common multiple of the integers m_x for all $x \in \Gamma_0$ and let $n_x = m/m_x$. Let $\bar{\Gamma}_0$ be the set of all ordered pairs $(x, i) \in \Gamma_0 \times \mathbb{Z}$ so that $1 \leqslant i \leqslant n_x$. We define $(x, i) \to (y, j)$ to be an arrow of $\bar{\Gamma}$ if $x \to y$ is an arrow of Γ and $i \equiv j$ modulo $\min(n_x, n_y)$. We define (x, i) to be projective if x is projective and we let $\tau(x, i) = (\tau x, i)$ if x is not projective.

LEMMA 8.1. *Let h: $\bar{\Gamma} \to \Gamma$ be the translation quiver morphism given by $h(x, i) = x$. Then h has the following properties*:

(1) *h maps projectives to projectives.*

(2) *For every $x \in \bar{\Gamma}_0$ and every $y \in (hx)^-$ we have that $x^- \cap h^{-1}(y)$ has exactly d'_{yhx} elements.*

(3) *For every $x \in \bar{\Gamma}_0$ and every $y \in (hx)^+$ we have that $x^+ \cap h^{-1}(y)$ has exactly d_{hxy} elements.*

Proof. Property (1) follows from the definition of $\bar{\Gamma}$. To verify property (2) let $x = (z, i)$. Then $x^- \cap h^{-1}(y) = \{(y, j) \mid 1 \leqslant j \leqslant n_y$ and $i \equiv j$ modulo $\min(n_z, n_y)\}$. Since Γ satisfies D we know that $\min(n_z, n_y)$ divides n_y and the quotient is d'_{yz}. But this is the number of integers modulo n_y congruent to any fixed integer 1 modulo $\min(n_z, n_y)$. The dual argument proves (3). ∎

LEMMA 8.2. *Let $(x, i) \in \bar{\Gamma}_0$ and give $\bar{\Gamma}$ the trivial valuation. Then the diagram of (x, i) is isomorphic to the diagram of x.*

Proof. Let $\bar{\mathcal{V}} = (\Sigma_{(x,i)})_0$, $\bar{\mathcal{A}} = (\Sigma_{(x,i)})_1$ and let $\bar{f}$: $(\Sigma_{(x,i)})_1 \to \bar{\Gamma}$ be the translation quiver morphism discussed in Section 6. We shall construct bijections $\bar{h}_j : \bar{\mathcal{V}}_j \to \mathcal{V}_j$ and $\bar{h}'_j : \bar{\mathcal{V}}'_j \to \mathcal{V}'_j$ so that $\coprod \bar{h}_j$ gives an isomorphism $\Sigma_{(x,i)} \to \Sigma_x$.

Since $\bar{\mathcal{V}}'_1 = \{(x, i)\}$ and $\mathcal{V}'_1 = \{x\}$ there is only one function $\bar{h}'_1$: $\bar{\mathcal{V}}'_1 \to \mathcal{V}'_1$. It follows from 8.1(3) that there exists a bijection $\bar{h}_0 : \bar{\mathcal{V}}_0 \to \mathcal{V}_0$ which makes the following diagram commute:

$$\begin{tikzcd} \bar{\mathcal{V}}_0 \arrow[r, "\bar{h}_0"] \arrow[d, "\bar{f}_0"'] & \mathcal{V}_0 \arrow[d, "f_0"] \\ \bar{\Gamma} \arrow[r, "h"] & \Gamma \end{tikzcd}$$

Suppose that $\bar{h}'_j$, $\bar{h}_{j-1}$, $\bar{h}_{j-2}$,... have been defined so that $\bar{h}'_j \amalg \bar{h}_{j-1} \amalg \cdots$ gives an isomorphism of translation quivers $\bar{\mathcal{V}}'_j \amalg \bar{\mathcal{V}}'_{j+1} \amalg \cdots \to \mathcal{V}'_j \amalg \mathcal{V}_{j-1} \amalg \cdots$ making the following diagram commute:

$$\begin{tikzcd} \bar{\mathcal{V}}'_j \amalg \bar{\mathcal{V}}_{j-1} \amalg \cdots \arrow[r, "\bar{h}'_j \amalg \cdots"] \arrow[d, "\bar{f}'_j \amalg \bar{f}_{j-1} \amalg \cdots"'] & \mathcal{V}'_j \amalg \mathcal{V}_{j-1} \amalg \cdots \arrow[d, "f'_j \amalg f_{j-1} \amalg \cdots"] \\ \bar{\Gamma} \arrow[r, "h"] & \Gamma \end{tikzcd}$$

Then by 8.1(2) $\bar{h}_j'$ can be extended to a bijection $\bar{h}_j\colon \bar{\mathcal{T}}_j \to \mathcal{T}_j$ so that $f_j\bar{h}_j = h\bar{f}_j$ and so that $\bar{h}_j \amalg \bar{h}_{j-1} \amalg \cdots$ is an isomorphism of translation quivers. By 8.1(1) the map $\bar{h}_{j+1}'\colon \bar{\mathcal{T}}_{j+1}' \to \mathcal{T}_{j+1}'$ uniquely determined by $\bar{h}_{j-1}$ is a bijection. ∎

THEOREM 8.3. *Let p be 0 or any prime. Then Γ satisfies A_p, B_p and C_p if and only if $\bar{\Gamma}$ does.*

Proof. This follows from 8.2 and the fact that A_p, B_p, C_p are conditions on diagrams. ∎

9. THE SET OF PRIMES

If Γ is a finite nonempty valued translation quiver satisfying condition D then let $\mathscr{P}(\Gamma)$ denote the set of all characteristics p so that Γ satisfies A_p, B_p, C_p. If Γ does not satisfy condition D then let $\mathscr{P}(\Gamma)$ be the empty set. The result of Theorem 8.3 can be stated as $\mathscr{P}(\Gamma) = \mathscr{P}(\bar{\Gamma})$. In this section we shall show that $\mathscr{P}(\Gamma)$ is either empty or contains all but a finite number of nonzero primes. The proof uses the universal coefficient theorem of algebraic topology.

If Γ is a finite translation quiver, $x, y \in \Gamma_0$ and R is a commutative ring let $C(x, y; R)$ be the free R-module generated by all paths $\alpha = \alpha_m\alpha_{m-1}\cdots\alpha_2\alpha_1$ from x to y so that each $\alpha_i \in \hat{\Gamma}_1$. This means that each α_i is either an arrow $\alpha_i\colon a \to b \in \Gamma_1$ or a "dotted arrow" $\alpha_i\colon \tau a \dashrightarrow a$. We define the bidegree of α to be the pair $(s + 2d, d)$, where s is the number of arrows in α and d is the number of dotted arrows in α. Let $C_m^j(x, y; R)$ be the submodule of $C(x, y; R)$ generated by paths with bidegree (j, m).

Let $C_m(x, y; R) = \coprod_{j \geqslant 2m} C_m^j(x, y; R)$ and let $\partial_m^i\colon C_m(x, y; R) \to C_{m-1}(x, y; R)$ be the degree 0 graded homomorphism which is defined on a path α by replacing the ith dotted arrow $\tau a \dashrightarrow a$ in α by the corresponding mesh which is the sum of all compositions of two arrows starting at τa and ending at a. Let $\partial_m = \sum_{i=1}^m (-1)^i \partial_m^i$. Then $(C_*(x, y; R), \partial_*)$ is a graded chain complex. Let $H_*(x, y; R)$ be the homology of this chain complex. Then the universal coefficient theorem gives us the following:

LEMMA 9.1. (1) $C_*(x, y; R) \cong C_*(x, y; \mathbb{Z}) \otimes R$.

(2) $H_0^j(x, y; R) \cong H_0^j(x, y; \mathbb{Z}) \otimes R$.

(3) $H_1^j(x, y; R) \cong H_1^j(x, y; \mathbb{Z}) \otimes R \amalg \operatorname{Tor}^{\mathbb{Z}}(H_0^j(x, y; \mathbb{Z}), R)$.

LEMMA 9.2. *$C_m^j(x, y; \mathbb{Z})$ and $H_m^j(x, y; \mathbb{Z})$ are finitely generated abelian groups for all x, y, m, j.*

LEMMA 9.3. *If k is a field and we give Γ the trivial k-modulation then $H_*(x, y) \cong H_*(x, y; k)$ for all $x, y \in \Gamma_0$.*

Let Γ be a finite nonempty translation quiver.

LEMMA 9.4. *The quiver Γ with the trivial k-modulation is an Auslander–Reiten quiver if and only if* char $k \in \mathscr{P}(\Gamma)$.

Proof. This follows from 7.2. ∎

LEMMA 9.5. *If $0 \notin \mathscr{P}(\Gamma)$ then $\mathscr{P}(\Gamma)$ is empty.*

Proof. Let k be any field of characteristic 0 and give Γ the trivial k-modulation. Since Γ is not Auslander–Reiten one of the conditions of 4.2 is not satisfied.

If 4.2(a_f) does not hold then from 9.2, 9.3 and 9.1(2) we conclude that $H_0^{4n-1}(x, y; \mathbb{Z})$ has an infinite cyclic summand for some $x, y \in \Gamma_0$ (n is the number of vertices of Γ). This implies that $H_0^{4n-1}(x, y; K) \neq 0$ for any field K.

If 4.2(b_f) does not hold then from 9.1(3) we conclude that $H_1^j(x, y; \mathbb{Z})$ contains an infinite cyclic summand for some $x, y \in \Gamma_0$ and $j \leqslant 4n$ so $H_1^j(x, y; K) \neq 0$ for any field K.

If 4.2(c_f') fails then we use the right exactness of $\otimes k$ to conclude that $\operatorname{coker}(H_0^j(p, i; \mathbb{Z}) \to H_0^{j+1}(p, ei^+; \mathbb{Z}))$ has an infinite cyclic summand so (c_f') fails for all primes. ∎

LEMMA 9.6. *If $0 \in \mathscr{P}(\Gamma)$ then $\mathscr{P}(\Gamma)$ contains all but a finite number of primes.*

Proof. We will show that there is only a finite number of primes for which the conditions of 4.2 fail.

Since (a_f) holds over fields of characteristic 0 we know that $H_0^{4n-1}(x, y; \mathbb{Z})$ is a finite abelian group. If the characteristic of a field k does not divide the order of any of these groups then (a_f) holds over k.

Since (b_f) holds over $\mathbb{Q}$ we know that $H_1^j(x, y; \mathbb{Z})$ is a finite abelian group for $j \leqslant 4n$. If the characteristic of the field k does not divide the orders of these groups or the orders of the torsion subgroups of the groups $H_0^j(x, y; \mathbb{Z})$ where $j \leqslant 4n$ then (b_f) holds over k.

Since (c_f') holds over $\mathbb{Q}$ we know that $\operatorname{coker}(H_0^j(p, i; \mathbb{Z}) \to H_0^{j+1}(p, ei^+; \mathbb{Z}))$ is a finite abelian group for $j \leqslant 4n-2$, p projective and i injective. If char k does not divide the orders of any of these groups then (c_f') holds over k. ∎

THEOREM 9.7. *Let Γ be a finite nonempty valued translation quiver which satisfies condition D. Then $\mathscr{P}(\Gamma)$ is either empty or contains all but a finite number of nonzero primes.*

Proof. By 8.3 we are reduced to the trivially valued case where we can apply 9.5 and 9.6. ∎

THEOREM 9.8. *Let Γ be a finite nonempty valued translation quiver satisfying condition D. Then $\mathscr{P}(\Gamma) = \mathscr{P}(\bar{\Gamma})$. In particular Γ is Auslander–Reiten if and only if $\bar{\Gamma}$ is the Auslander–Reiten quiver of an artin algebra over $\mathbb{C}$ (the complex numbers).*

Proof. The equation $\mathscr{P}(\Gamma) = \mathscr{P}(\bar{\Gamma})$ is a restatement of 8.3. This implies (Γ is Auslander–Reiten) $\Leftrightarrow$ $(\mathscr{P}(\Gamma) \neq \varnothing)$ $\Leftrightarrow$ $(\mathscr{P}(\bar{\Gamma}) \neq \varnothing)$ $\Leftrightarrow$ $(0 \in \mathscr{P}(\bar{\Gamma}))$ $\Leftrightarrow$ ($\bar{\Gamma}$ with the trivial $\mathbb{C}$-modulation is Auslander–Reiten). ∎

10. COVERINGS

In this section we develop basic properties of coverings and show that if $\tilde{\Gamma} \to \Gamma$ is a covering map of finite quivers then Γ is an Auslander–Reiten quiver if and only if $\tilde{\Gamma}$ is. We also write down the proofs of the accepted results about universal coverings in general.

DEFINITION 10.1. A translation quiver morphism $f: \tilde{\Gamma} \to \Gamma$ is called a *covering* if the following conditions are satisfied:

(1) $f: \tilde{\Gamma}_0 \to \Gamma_0$ is an epimorphism.

(2) $f(x)$ is projective if and only if x is projective. $f(x)$ is injective if and only if x is injective.

(3) For every $x \in \Gamma_0$, f induces bijections $x^- \to (fx)^-$ and $x^+ \to (fx)^+$.

DEFINITION 10.2. A morphism (f, f_*, ϕ): $\tilde{\Gamma} \to \Gamma$ of k-modulated translation quivers is called a *covering morphism* if f is a covering of translation quivers and f_* is an isomorphism on all division algebras and bimodules.

LEMMA 10.3. *Let (f, f_*, ϕ): $\tilde{\Gamma} \to \Gamma$ be a covering morphisms and let (f', f'_*, ϕ'): $\Gamma' \to \Gamma$ be any morphism. If f' lifts to a translation quiver morphism h: $\Gamma' \to \tilde{\Gamma}$ then (f', f'_*, ϕ') lifts to a morphism (h, h_*, ψ): $\Gamma' \to \tilde{\Gamma}$, where h_*, ψ are uniquely determined by h.*

Proof. Since f_* is always an isomorphism we must have $h_* = f_*^{-1} f'_*$ and $\psi x = f_*^{-1}(\phi' x (\phi h x)^{-1})$. ∎

PROPOSITION 10.4. *Let Γ be a k-modulated translation quiver and let f: $\tilde{\Gamma} \to \Gamma$ be a covering of the underlying translation quiver. Then $\tilde{\Gamma}$ admits a k-*

modulation, which is unique up to isomorphism, so that f extends to a covering morphism (f, f_*, ϕ).

Proof. Uniqueness follows from 10.3. To show existence let:

(a) $Fx = Ffx$ for every $x \in \tilde{\Gamma}_0$.

(b) ${}_yM_x = {}_{fy}M_{fx}$ for every $x \to y \in \tilde{\Gamma}_1$.

(c) For every nonprojective $x \in \tilde{\Gamma}_0$ let $\tilde{\tau}_* : Fx \to F\tau x$ be given by $Fx = Ffx \to^{\tau_*} F\tau fx = F\tau x$.

(d) For every $y \to x \in \tilde{\Gamma}_1$ with nonprojective x let $\tilde{\sigma}_*$ be given as follows: ${}_yM_{\tau x} \otimes_{F\tau x} {}_xM_y = {}_{fy}M_{\tau fx} \otimes_{F\tau fx} {}_{fx}M_{fy} \to^{\sigma_*} Ffy = Fy$.

This modulation of $\tilde{\Gamma}$ will be called the *induced modulation.* When $\tilde{\Gamma}$ is modulated in this way the covering f extends to the covering morphism $(f, 1, 1)$: $\tilde{\Gamma} \to \Gamma$. ∎

Remark 10.5. Proposition 10.4 implies that modulated translation quivers admit universal coverings because the universal covering of the underlying translation quiver can be modulated.

Remark 10.6. The fundamental group of a connected Auslander–Reiten quiver is free, since the Auslander–Reiten quiver of an artin algebra of finite representation type is tree finite (see Section 13).

LEMMA 10.7. *Let (f, f_*, ϕ): $\tilde{\Gamma} \to \Gamma$ be a covering morphism and let $H_*(f)$: $H_*(\tilde{\Gamma}) \to H_*(\Gamma)$ be the induced functor. Then given any $x \in \Gamma_0$ and $y \in \tilde{\Gamma}_0$ we get an isomorphism $H_*(f^{-1}x, y) \cong H_*(x, fy)$ induced by $H_*(f)$, where $f^{-1}x$ is the direct sum of all the elements of $f^{-1}(x)$.*

Proof. Consider the chain map $C^i_*(f^{-1}x, y) \to C^i_*(x, fy)$ induced by $C_*(f)$. We shall show by induction on i that these are isomorphisms. If $i \leqslant 0$ this map is easily seen to be an isomorphism. So suppose $i \geqslant 1$. Then we have:

$$C^i_m(f^{-1}x, y) = \coprod_{z \in y^-} {}_yM_z \otimes_{Fz} C^{i-1}_m(f^{-1}x, z) \amalg C^{i-2}_{m-1}(f^{-1}x, \tau y)$$

$$\cong \coprod_{z \in y^-} {}_{fy}M_{fz} \otimes_{Ffz} C^{i-1}_m(x, \tau fy) = C^i_m(x, fy). \quad ∎$$

THEOREM 10.8. *Let $\tilde{\Gamma} \to \Gamma$ be a covering morphism of finite k-modulated translation quivers. Then Γ is an Auslander–Reiten quiver if and only if $\tilde{\Gamma}$ is an Auslander–Reiten quiver.*

Proof. Lemma 10.7 implies that Γ satisfies conditions (a) and (b) of 3.4

if and only if $\tilde{\Gamma}$ does. The following commutative diagram shows that Γ satisfies the dual of 3.4(c) as given in 3.5(3) if and only if $\tilde{\Gamma}$ does.

$$\begin{array}{ccc} H_0(f^{-1}p, i) & \longrightarrow & H_0(f^{-1}p, ei^+) \\ \downarrow \cong & & \downarrow \cong \\ H_0(p, fi) & \longrightarrow & H_0(p, efi^+) \end{array} \quad \blacksquare$$

Remark 10.9. By 10.6 and 10.8 every finite nonsimply connected Auslander–Reiten quiver has an infinite number of finite coverings which are Auslander–Reiten quivers.

11. Standard Algebras

Standard algebras are determined by their Auslander–Reiten quivers. The purpose of this section is to prove that every finite Auslander–Reiten quiver has a finite covering which is standard.

DEFINITION 11.1. A *standard algebra* is an artin algebra Λ of finite representation type so that $\operatorname{ind}\Lambda \cong H_0(\Gamma_\Lambda)$.

DEFINITION 11.2. Let Λ be a standard algebra. If there are no nonstandard algebras whose Auslander–Reiten quivers are finite coverings of Γ_Λ we say that Γ_Λ is a *standard* Auslander–Reiten quiver. (Note that Γ_Λ is a finite covering of itself.)

PROPOSITION 11.3. *Let Λ be an artin algebra of finite representation type. Then Λ is standard if and only if Λ is isomorphic to the graded algebra* $\operatorname{End}(G(\ ,\Lambda))$.

Proof. Suppose that Λ is a standard algebra. Then $\Lambda \cong \operatorname{End}(\Lambda) \cong \operatorname{End}((\ ,\Lambda)) \cong \operatorname{End}(H_0(\ ,\Lambda)) \cong \operatorname{End}(G(\ ,\Lambda))$ (the last isomorphism follows from Lemma 3.1). Conversely, from the proof of 3.4 it follows that $H_0(\Gamma_\Lambda) \cong \operatorname{ind} H_0(\Lambda, \Lambda)$. But $H_0(\Lambda, \Lambda) \cong \operatorname{End}(H_0(\ ,\Lambda)) \cong \operatorname{End}(G(\ ,\Lambda)) \cong \Lambda$. Thus Λ is standard. $\blacksquare$

LEMMA 11.4. *Suppose that Γ is a finite Auslander–Reiten quiver so that for every pair of vertices $x, y \in \Gamma_0$ there is at most one integer i so that $H_0^i(x, y) \neq 0$. Then Γ is standard.*

Proof. Suppose that $\Gamma \cong \Gamma_\Lambda$. Then for any two indecomposable Λ-modules x and y we have $G(x, y) \cong H_0(x, y) = H_0^i(x, y) \cong G_i(x, y)$, so

$(x, y) = G(x, y)$. Thus $\operatorname{ind} \Lambda \cong H_0(\Gamma_\Lambda)$, i.e., Λ is standard. By 4.5 every finite covering of Γ satisfies the same condition as Γ, so Γ is standard.∎

THEOREM 11.5. *Let Γ be a finite Auslander–Reiten quiver. Then Γ has a finite covering which is standard.*

Proof. We will assume that Γ is connected. (If Γ is not connected we apply the following argument to each component.)

Let $x_0 \in \Gamma_0$ and let ψ: $\Pi_1(\Gamma, x_0) \to \mathbb{Z}$ be the homomorphism which takes the homotopy class of a loop $\alpha = \alpha_1 \alpha_2 \cdots \alpha_j$ to the integer $\psi[\alpha] = a + 2b - c - 2d$, where

$$a = \# \text{ of } \alpha_i'\text{s in } \Gamma_1,$$

$$b = \# \text{ of } \alpha_i'\text{s which are dotted arrows } \tau x \dashrightarrow x,$$

$$c = \# \text{ of } \alpha_i'\text{s in } \Gamma_1^{\text{op}},$$

$$d = \# \text{ of } \alpha_i'\text{s which are inverse dotted arrows } \tau x \dashleftarrow x.$$

Let $\tilde{\Gamma}$ be a finite covering of Γ so that the image of the composition

$$\Pi_1(\tilde{\Gamma}, \tilde{x}_0) \longrightarrow \Pi_1(\Gamma, x_0) \xrightarrow{\psi} \mathbb{Z} \tag{$*$}$$

does not contain the integers $1, 2, \dots, 4n - 2$, where n is the number of vertices of Γ and $\tilde{x}_0$ is any vertex of $\tilde{\Gamma}$ which maps to x_0. We claim that $\tilde{\Gamma}$ satisfies the condition of 11.4 and is therefore standard. We can use 10.4 to show that such a $\tilde{\Gamma}$ always exists.

Since $\tilde{\Gamma}$ is a covering of Γ we know by Theorem 4.2 and 10.7 that $H_0^{4n-1} = 0$ on $\tilde{\Gamma}$. Let $x, y \in \tilde{\Gamma}_0$. If i, j are two distinct integers so that $H_0^i(x, y) \neq 0$ and $H_0^j(x, y) \neq 0$ then we have two paths ω_i and ω_j of lengths i and j from x to y. If β is any path from $\tilde{x}_0$ to x we get a loop $\beta^{-1}\omega_j^{-1}\omega_i\beta$ representing an element of $\Pi_1(\tilde{\Gamma}, \tilde{x}_0)$ which goes to $i - j$ under the composition $(*)$. Since $0 \leqslant i, j \leqslant 4n - 2$ this is a contradiction to the construction used to construct $\tilde{\Gamma}$. ∎

Remark 11.6. This theorem implies that covering algebras do not exist in general since every standard algebra contains a prime field and, for example, there are no ring homomorphisms from any $\mathbb{Z}_2$-algebra to any $\mathbb{Z}_4$-algebra.

12. REMARKS

This section contains two ideas which M. Auslander encouraged us to add to this paper.

(a) We show that every finite modulated Auslander–Reiten quiver admits a finite covering without short chains.

(b) We show that any automorphism of an artin algebra of finite representation type which fixes one module fixes a projective.

DEFINITION 12.1. Let Λ be an artin algebra. Then $A \to^f B \to^g \mathrm{DTr}\, A$ is called a *short chain* of Λ-modules if A and B are indecomposable Λ-modules with A nonprojective and f, g are nonzero Λ-morphisms.

DEFINITION 12.2. Let Γ be a modulated translationquiver. Then $a \to^f b \to^g \tau a$ is a *short chain* in Γ if $a, b \in \Gamma_0$ with a nonprojective and f, g are nonzero morphism in the mesh category $H_0(\Gamma)$.

PROPOSITION 12.3. *Let Λ be an artin algebra of finite representation type. Then Λ has a short chain if and only if Γ_Λ does.*

Proof. Let A, B be nonzero indecomposable Λ-modules and let a, b be the corresponding vertices of Γ_Λ. Then by 3.1 we have $H_0(a, b) \cong G(A, B)$. Thus $H_0(a, b) \neq 0$ if and only if $(A, B) \neq 0$. Consequently a short chain $a \to b \to \tau a$ would give a short chain $A \to B \to \mathrm{DTr}\, A$ and vice versa. ∎

THEOREM 12.4. *Let Γ be a finite modulated Auslander–Reiten quiver. Then Γ has a finite covering which has no short chains.*

Proof. Since each component of Γ is Auslander–Reiten we may assume that Γ is connected. Let n be the number of vertices of Γ, let x_0 be any vertex of Γ and let ψ: $\pi_1(\Gamma, x_0) \to \mathbb{Z}$ be the homomorphism described in the proof of 11.5. Let $\tilde{\Gamma}$ be a finite covering of Γ so that the image of the composition $\pi_1(\tilde{\Gamma}, x_0) \to \pi_1(\Gamma, x_0) \to \mathbb{Z}$ does not contain the integers 1, 2,..., $8n-2$. Then $\tilde{\Gamma}$ has no short chains. If $\tilde{\Gamma}$ had a short chain $a \to b \to \tau a$ then there would be a path from a to b consisting of $4n-2$ or fewer arrows (since $H_0^{4n-1} = 0$ on $\tilde{\Gamma}$ as in the proof of 11.5.) Similarly there would be a path from b to τa consisting of $\leqslant 4n-2$ arrows. We know that we can get from τa to a with two arrows so we have a loop consisting of m arrows where $2 \leqslant m \leqslant 8n-2$. The integer m lies in the image of $\pi_1(\tilde{\Gamma}, x_0) \to \pi_1(\Gamma, x_0) \to \mathbb{Z}$. ∎

We give an example of how the associated unvalued quiver $\bar{\Gamma}$ can be used to reduce problems to the algebraically closed case.

THEOREM 12.5. *Let Λ be an artin algebra of finite representation type and let ϕ be an automorphism of Λ which fixes the isomorphism class of some nonzero indecomposable Λ-module M. Then ϕ fixes the isomorphism class of a nonzero indecomposable projective.*

Proof. We use the fact that this is true if Λ is a finite-dimensional algebra over $\mathbb{C}$. (It was shown in [6] that this result holds for finite-dimensional algebras over algebraically closed fields.) We may assume that Λ is indecomposable since ϕ fixes the component of Λ corresponding to M.

Let Γ be the Auslander–Reiten quiver of Λ considered as a valued quiver. Then ϕ induces an automorphism ϕ_* of Γ which fixes one vertex x_0 corresponding to M. Let $\bar{\Gamma}$ be the associated unvalued quiver of Γ and let $\bar{\phi}_*$ be the automorphism of $\bar{\Gamma}$ given by $\bar{\phi}_*(x, i) = (\phi_*(x), i)$.

By 9.8 we know that $\bar{\Gamma}$ with the trivial $\mathbb{C}$-modulation is an Auslander–Reiten quiver. The automorphism $\bar{\phi}_*$ induces an automorphism of the associated $\mathbb{C}$-algebra of finite representation type which fixes one module and thus fixes a projective. This means that $\bar{\phi}_*$ fixes a projective vertex (p, i) which implies that p is a projective vertex of Γ left fixed by ϕ_*. This corresponds to an indecomposable projective left fixed by ϕ. ∎

13. Appendix

In this section we give a short elementary proof of the well-known fact that Auslander–Reiten quivers are tree finite. The proof does not depend on the results of this paper or our previous paper [5].

Definition 13.1. A *sectional path* in mod Λ is defined to be a sequence of nonzero indecomposable Λ-modules $X_0, X_1, ..., X_n$ and irreducible maps f_i: $X_{i-1} \to X_i$ for $1 \leqslant i \leqslant n$ such that $X_{i-2} \not\cong \mathrm{DTr}\, X_i$ for $2 \leqslant i \leqslant n$.

Lemma 13.2. *Let Λ be any artin algebra and let $X = X_0 \to^{f_1} X_1 \to \cdots \to^{f_n} X_n = Y$ be a sectional path. Then there do not exist maps g: $X \to Z$ and f'_n: $Z \to Y$ such that (f'_n, f_n): $Z \amalg X_{n-1} \to Y$ is a right minimal almost split map and that the image of $(\ , f'_n\, g + f_n \cdots f_1)$ is contained in $r^{n+1}(\ , Y)$.*

Proof. Proof is by induction. If $n = 1$ the statement is obviously true so suppose that $n \geqslant 2$ and the lemma is true for sectional paths of length $n - 1$. Suppose that it fails for the sectional path $f_n \cdots f_1$. Then let g, f'_n be as given in the statement of the lemma and let

$$0 \longrightarrow K \xrightarrow{(k', k)} Z \amalg X_{n-1} \xrightarrow{(f'_n, f_n)} Y$$

be an exact sequence. Then K is either 0 or $\mathrm{DTr}\, Y$ so (k, f_{n-1}): $K \amalg X_{n-2} \to X_{n-1}$ is an irreducible map.

Since $\mathrm{im}(\ , f'_n g + f_n \cdots f_1) \subset r^{n+1}(\ , Y)$ there is a map (h', h): $X \to Z \amalg X_{n-1}$ which is a sum of chains of at least n irreducible maps such that

$(f'_n, f_n)(h', h) = f'_n g + f_n \cdots f_1 = (f'_n, f_n)(g, f_{n-1} \cdots f_1)$. This means that $(h', h) - (g, f_{n-1} \cdots f_1)$ factors through $K = \ker(f'_n, f_n)$. Thus there is a map $g'\colon X \to K$ so that $(k', k)g' = (h', h) - (g, f_{n-1} \cdots f_1)$. Therefore $h = kg' + f_{n-1} \cdots f_1$. But $\operatorname{im}(\ , h) \subset r^n(\ , X_{n-1})$ which contradicts the induction hypothesis. ∎

THEOREM 13.3. *Let Λ be any artin algebra and let $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n = Y$ be a sectional path. Then* $\operatorname{im}(\ , f_n \cdots f_1) \subset r^n(\ , Y)$ *and* $\operatorname{im}(\ , f_n \cdots f_1) \not\subset r^{n+1}(\ , Y)$.

Proof. Suppose that $\operatorname{im}(\ , f_n \cdots f_1) \subset r^{n+1}(\ , Y)$. Then we complete f_n to a right minimal almost split map $(f'_n, f_n)\colon Z \amalg X_{n-1} \to X_n$ and let $g\colon X \to Z$ be the zero map. By Lemma 13.1 this gives a contradiction. ∎

COROLLARY 13.4. *The composition of irreducible maps on a sectional path is nonzero.*

COROLLARY 13.5. *If Λ is of finite representation type then all sectional paths are finite, in particular it is tree finite.*

COROLLARY 13.6. *Let Γ be a finite connected Auslander–Reiten quiver. Then $\pi_1 \Gamma$ is a free group.*

Proof. This follows from 13.5 and [4, Theorem 4.2]. ∎

COROLLARY 13.7 [3]. *Let Λ be any artin algebra. Then there does not exist a sectional path $X_0 \to X_1 \to \cdots \to X_n$ such that $X_0 = X_{n-1}$ and $X_1 = X_n$. (There are no "sectional cycles.")*

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